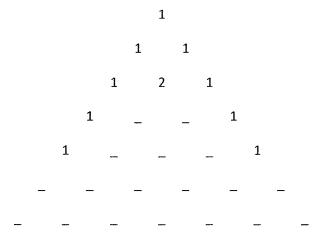
The Binomial Theorem

Pascal's Triangle

Pascal's Triangle is a triangular array of numbers named after the French mathematician Blaise Pascal (1623-1662). The first and last numbers of each row of Pascal's triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number.



Binomial expansion

Pascal noticed that numbers in this triangle are precisely the same numbers that are the coefficients of the binomial expansions:

$$(a+b)^{0} = 1$$

$$(a+b)^{1} = 1a+1b$$

$$(a+b)^{2} = 1a^{2} + 2ab + 1b^{2}$$

$$(a+b)^{3} = 1a^{3} + 3a^{2}b + 3ab^{2} + 1b^{3}$$

$$(a+b)^{4} = \underline{\qquad} a^{4} + \underline{\qquad} a^{3}b + \underline{\qquad} a^{2}b^{2} + \underline{\qquad} ab^{3} + \underline{\qquad} b^{4}$$

$$(a+b)^{5} = \underline{\qquad} + \underline{\qquad} +$$

The top row of Pascal's triangle is called the ______th row because it corresponds to the binomial expansion $(a+b)^0=1$. The next row is called the ______ row because it corresponds to the binomial expansion $(a+b)^1=1a+1b$. In general, the nth row in Pascal's Triangle gives the coefficients of $(a+b)^n$.

Watch for these characteristics as we do the following problems:

- The number of terms is one more than the power of the expansion, n.
- The coefficient of the first term and the last term is <u>always one</u> (before we simplify)
- The coefficient of the second term and the next to the last term is always n (before we simplify)
- The sum of the exponents in each term is n (before we simplify)

Expand the binomial using Pascal's Triangle and simplify the expression:

$$(x+2)^5$$

Expand the binomial using Pascal's Triangle and simplify the expression:

$$(x-y)^3$$

Write the first three terms of the binomial expansion using Pascal's Triangle and simplify the expression:

$$(3x^2-5y)^6$$

How many terms will be in the binomial expansion $(2x+3y)^{10}$?

Write the first two terms of the expansion.

How many terms will be in the binomial expansion $(x^{1/2} + 1)^{30}$?

Write the last two terms of the expansion.

Homework:

Use Pascal's Triangle to expand the expression:

1.
$$(2x-3y)^3$$

$$2. \qquad \left(x+2y\right)^4$$

$$3. \qquad \left(1 + \frac{1}{x}\right)^6$$

- 4. Find the last two terms in the expansion $\left(a^{2/3}+a^{1/3}\right)^{25}$
- 5. If $f(x) = x^4$, find the difference quotient $\frac{f(x+h) f(x)}{h}$
- 6. If $f(x) = x^6$, find the difference quotient $\frac{f(x+h) f(x)}{h}$.

If you substitute 0 for h in your answer, what is your answer? We will discuss the significance of this problem in class.

Answers are on the back.

Answers

1.
$$8x^3 - 36x^2y + 54xy^2 - 27y^3$$

2.
$$x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$$

3.
$$1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}$$

4.
$$25a^{26/3} + a^{25/3}$$

5.
$$4x^3 + 6x^2h + 4xh^2 + h^3$$

6.
$$6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5$$

If
$$h = 0$$
, we get $6x^5$.

Some new notation: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

We say "n choose r"

Try it out:

$$\binom{7}{0} = \frac{7!}{0!(7-0)!} =$$

$$\binom{7}{1} = \frac{7!}{1!(7-1)!} =$$

$$\binom{7}{2}$$
=

$$\binom{7}{3}$$
=

$$\binom{7}{4}$$
=

$$\binom{7}{5}$$
=

$$\binom{7}{6}$$

$$\binom{7}{7}$$

Compare these values to the 7th line of Pascal's Triangle:

The Binomial Theorem

$$(a+b)^{n} = \binom{n}{0}a^{n} + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^{2} + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^{n} = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^{r}$$

In general, the r+1 term of the series will be given by $\binom{n}{r}a^{n-r}b^r$.

Notice that the lower value in the coefficient $\binom{n}{r}$ matches the exponent of the second term.

Find the 7th term of the expansion $(a+b)^{30}$:

Find the term that contains x^5 in the expansion $(2x+y)^{20}$:

Homework

- 1. Use the Binomial Theorem to find the fifth term in the expansion $(x-1)^9$
- 2. Use the Binomial Theorem to find the term containing y^{14} in the expansion $\left(x^2+y\right)^{22}$
- 3. If $f(x) = x^4$, find the difference quotient $\frac{f(x+h) f(x)}{h}$

Answers

1.
$$126x^5$$

2.
$$319,770x^{16}y^{14}$$

3.
$$4x^3 + 6x^2h + 4xh^2 + h^3$$