The Binomial Theorem

Pascal's Triangle

Pascal's Triangle is a triangular array of numbers named after the French mathematician Blaise Pascal (1623-1662). The first and last numbers of each row of Pascal's triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & \_ & & \_ & & 1 & & & \\
 & & 1 & & \_ & & \_ & & \_ & & 1 & & \\
 & \_ & & \_ & & \_ & & \_ & & \_ & & \_ & \\
\_ & & \_ & & \_ & & \_ & & \_ & & \_ & & \_
\end{array}
$$

Binomial expansion

Pascal noticed that numbers in this triangle are precisely the same numbers that are the coefficients of the binomial expansions:

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = \_\_\_\, a^4 + \_\_\_\, a^3b + \_\_\_\, a^2b^2 + \_\_\_\, ab^3 + \_\_\_\, b^4$$

$$(a+b)^5 = \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_$$

The top row of Pascal's triangle is called the __________th row because it corresponds to the binomial expansion $(a+b)^0 = 1$. The next row is called the ______________ row because it corresponds to the binomial expansion $(a+b)^1 = 1a + 1b$. In general, the nth row in Pascal's Triangle gives the coefficients of $(a+b)^n$.

Watch for these characteristics as we do the following problems:

- The number of terms is one more than the power of the expansion, n.
- The coefficient of the first term and the last term is always one (before we simplify)
- The coefficient of the second term and the next to the last term is always n (before we simplify)
- The sum of the exponents in each term is n (before we simplify)

Expand the binomial using Pascal's Triangle and simplify the expression:

$(x+2)^5$

Expand the binomial using Pascal's Triangle and simplify the expression:

$(x-y)^3$

Write the first three terms of the binomial expansion using Pascal's Triangle and simplify the expression:

$(3x^2-5y)^6$

How many terms will be in the binomial expansion $(2x+3y)^{10}$?

Write the first two terms of the expansion.

How many terms will be in the binomial expansion $(x^{1/2}+1)^{30}$?

Write the last two terms of the expansion.

Homework:

Use Pascal's Triangle to expand the expression:

1. $(2x-3y)^3$

2. $(x+2y)^4$

3. $\left(1+\frac{1}{x}\right)^6$

4. Find the last two terms in the expansion $\left(a^{2/3}+a^{1/3}\right)^{25}$

5. If $f(x)=x^4$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

6. If $f(x)=x^6$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$.

   If you substitute 0 for h in your answer, what is your answer? We will discuss the significance of this problem in class.

Answers are on the back.

Answers

1. $8x^3 - 36x^2y + 54xy^2 - 27y^3$

2. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

3. $1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}$

4. $25a^{26/3} + a^{25/3}$

5. $4x^3 + 6x^2h + 4xh^2 + h^3$

6. $6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5$

   If $h = 0$, we get $6x^5$.

Some new notation: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ We say " n choose r "

Try it out:

$$\binom{7}{0} = \frac{7!}{0!(7-0)!} =$$

$$\binom{7}{1} = \frac{7!}{1!(7-1)!} =$$

$$\binom{7}{2} =$$

$$\binom{7}{3} =$$

$$\binom{7}{4} =$$

$$\binom{7}{5} =$$

$$\binom{7}{6} =$$

$$\binom{7}{7} =$$

Compare these values to the 7th line of Pascal's Triangle:

_____ _____ _____ _____ _____ _____ _____ _____

The Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

In general, the $r+1$ term of the series will be given by $\begin{pmatrix} n \\ r \end{pmatrix} a^{n-r} b^r$.

Notice that the lower value in the coefficient $\begin{pmatrix} n \\ r \end{pmatrix}$ matches the exponent of the second term.

Find the 7$^{\text{th}}$ term of the expansion $(a+b)^{30}$:

Find the term that contains x^5 in the expansion $(2x+y)^{20}$:

Homework

1. Use the Binomial Theorem to find the fifth term in the expansion $(x-1)^9$

2. Use the Binomial Theorem to find the term containing y^{14} in the expansion $\left(x^2+y\right)^{22}$

3. If $f(x)=x^4$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Answers

1. $126x^5$

2. $319,770x^{16}y^{14}$

3. $4x^3 + 6x^2h + 4xh^2 + h^3$